Lecture 7

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1 Matrix equations and the inverse

1.1 Algorithm of solving

Suppose we have 2 given matrices A and B, and we'd like to find a matrix X such that

$$AX = B. \tag{1}$$

Note, that B should not be a square matrix, for example in case of a linear system B is a matrix with only 1 column, and X is a matrix with 1 column as well. Suppose matrix A is invertible, i.e. there exists an inverse A^{-1}. So, we can multiply both parts of the equation (1) by A^{-1} from the left side. We'll get: $A^{-1} \cdot AX = A^{-1}B \Longrightarrow (A^{-1}A)X = A^{-1}B \Longrightarrow X = A^{-1}B$. So, the solution of this equation in case when A is invertible is

$$X = A^{-1}B. \tag{2}$$

Now let's suppose we need to find the inverse of any square matrix A. If matrix X is an inverse for A then it satisfies the following equation: $AX = I$, where I is an identity matrix. It is a special case of the equation (1). So we see, that finding an inverse is a special case of solving the matrix equation (1).

We'll provide an algorithm of solving the general case of matrix equation (1).

Algorithm of solving the equation $\boldsymbol{AX = B}$.

Step 1 Construct the augmented matrix from A and B writing the matrix B to the right of matrix A, i.e. if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix},$$

then augmented matrix is

$$(A|B) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & b_{11} & b_{12} & \dots & b_{1m} \\ a_{21} & a_{22} & \dots & a_{2n} & b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_{n1} & b_{n2} & \dots & b_{nm} \end{array}\right) \tag{3}$$

Step 2 Perform elementary row operations to reduce the matrix A in the left side of the augmented matrix to its RREF. (But perform elementary row operations with whole rows of $(A|B)$!) If we get a zero-row in A-half of the matrix $(A|B)$, then the matrix A is not invertible, and we should stop our algorithm. Otherwise, the RREF of the matrix A will be equal to I (we will prove this fact in the next part). So, by the end of this step we'll have a matrix I in the left half of the augmented matrix. Than the matrix from the right half will be $A^{-1}B$.

In case of finding the inverse, we should use augmented matrix of the form $(A|I)$, and after reducing A to its RREF, which in case of invertible matrix should be equal to I, we'll get A^{-1} in the right half of the augmented matrix.

Example 1.1. *Let's find the inverse of the following matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$$

The augmented matrix will be

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$$

Now, subtract the 1st line from the 1nd one and the 3rd one. We'll have:

$$A = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right)$$

Then we should subtract the 2nd row from the 3rd and we'll reduce the left side of augmented matrix to REF (not RREF yet!):

$$A = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

Now, we'll reduce the left side of augmented matrix to the RREF. We'll use backward elimination. So, we subtract the 3rd line multiplied by 2 from the 2nd one and subtract it from the 1st:

$$A = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

Now, we'll subtract the 2nd line from the 1st one and the left side of augmented matrix will be in its RREF — it will be equal to the identity matrix:

$$A = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

So, the inverse for A will be the following:

$$A = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

Now by checking we can see that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.2 Discussion of the algorithm

In this part we will discuss some issues related to the equation

$$AX = B \tag{4}$$

and to the concept of the inverse. It may happen that if A is not invertible, this equation has no solutions or infinitely many solutions. The following examples demonstrate it.

Example 1.2. *Consider the equation*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Obviously matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible, but any 2×2 matrix X is a solution for this equation.

Example 1.3. *Consider the equation*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It was shown before that matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *is not invertible, and so this equation has no solution.*

To proceed we'll need to see what happens with the matrix if we reduce it to its RREF by elementary row operations. Applying an elementary row operation is equivalent to multiplying by the appropriate type of special matrices. So, suppose we applied s elementary row operations with corresponding matrices $E_1, E_2, \dots, E_s$ to the matrix A. After the first operation we get E_1A, after the second — $E_2(E_1A)$, etc., and after applying all of them the result will be $E_s(E_{s-1}(\cdots E_2(E_1A)\cdots))$.

Moreover, we'll need the following lemma.

Lemma 1.4. *If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof. It is sufficient to check that $(AB)\cdot(B^{-1}A^{-1}) = I$. It is true, since

$$(AB)\cdot(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

□

From this lemma it follows that if $A_1, A_2, \dots, A_n$ are invertible matrices of the same size, then their product $A_1A_2\cdots A_n$ is invertible and $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\cdots A_1^{-1}$.

Another fact we will state in the following lemma:

Lemma 1.5. *The matrices of the elementary row operations are invertible.*

Proof. Using the notations from the previous lecture we can specify the inverses for the matrices, corresponding to elementary row operations:

$$P_{ij}^{-1} = P_{ij}; \tag{5}$$

$$Q_i(c)^{-1} = Q_i\left(\frac{1}{c}\right); \tag{6}$$

$$(I + cI_{ij})^{-1} = I - cI_{ij}. \tag{7}$$

It is easy to check that these identities are correct. □

Now we're ready to prove that the algorithm gives us the solution to the equation $AX = B$ in case if A is invertible. We'll state this result as the following lemma.

Lemma 1.6. *1. If the square matrix A is invertible, then its RREF is the identity matrix.*

2. If we can reduce the matrix A by elementary row operations to the identity matrix, i.e. if its RREF is identity matrix, then this algorithm gives us $A^{-1}B$ in the right half of the augmented matrix.

So, this lemma tells us that in case of the invertible matrix A the algorithm will not stop since we'll not get zero rows and finally we'll get identity matrix in the left part of the augmented matrix, and it will give us the correct solution of the equation (1) in the right half.

Proof of part 1. Suppose A is invertible, and its RREF is equal to some matrix B (which is in its own RREF). Then there exist matrices of elementary row operations E_1, E_2, $\ldots$, E_s such that $E_s \ldots E_2 E_1 A = B$. Since A is invertible and E_i's are invertible, B is also invertible. But if $B \neq I$ then B has a zero row, and B is not invertible, which contradicts with what we proved before (that B is invertible). So, $B = I$. $\square$

Proof of part 2. So, by applying elementary row operations to the augmented matrix, we multiply its parts by matrices of elementary row operations, i.e. during this algorithm we multiply A and B by elementary matrices E_1, E_2, $\ldots$, E_s. So, by the end of our algorithm augmented matrix has the following form:

$$(E_s \ldots E_2 E_1 A | E_s \ldots E_2 E_1 B) = (I|C).$$

So, we have that $(E_s \cdots E_2 E_1)A = I$, so that $(E_s \cdots E_2 E_1) = A^{-1}$, and so $C = (E_s \cdots E_2 E_1)B = A^{-1}B$ – what we wanted to get. $\square$